MORE APPROXIMATION ON DISKS

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Abstract. In this paper we study the function algebra generated by z^2 and g^2 on a small closed disk centered at the origin of the complex plane. We prove, using a biholomorphic change of coordinates and already developed techniques in this area, that for a large class of functions g this algebra consists of all continuous functions on the disk.

1. Introduction

Let g be a C^1 function defined in a neighbourhood of the origin in the complex plane, with $g(0) = 0$, $g_z(0) = 0$, $g_{\bar{z}}(0) = 1$ (i.e. g behaves like $\bar{z}$ near 0), and such that z^2 and g^2 separate points near 0. Is it possible to find a small closed disk D about 0 in the complex plane, so that every continuous function on D can be approximated uniformly on D by polynomials in z^2 and g^2? In other words is the function algebra $[z^2, g^2; D]$ on D generated by z^2 and g^2, i.e. the uniform closure in $C(D)$ of the polynomials in z^2 and g^2, equal to $C(D)$? It has been shown that both answers *no* and *yes* are possible, cf. [8, 5].

The motivating question for this approximation problem was whether $[z^2, \bar{z}^2 + \bar{z}^3; D]$ equals $C(D)$. The answer has been given recently by O'Farrell and Sanabria-García and is *no*, cf. [6].

The crucial point in showing whether or not the algebra $[z^2, g^2; D]$ coincides with $C(D)$, is to determine whether or not the preimage of $X = (z^2, g^2)(D)$ under the map $\Pi(\zeta_1, \zeta_2) = (\zeta_1^2, \zeta_2^2)$ is polynomially convex. Now the set $\Pi^{-1}(X)$ consists of the following four disks:

$$\begin{aligned}
D_1 &= \{(z, g(z)) : z \in D\},\\
D_2 &= \{(-z, -g(z)) : z \in D\} = \{(z, -g(-z)) : z \in D\},\\
D_3 &= \{(-z, g(z)) : z \in D\},\\
D_4 &= \{(z, -g(z)) : z \in D\} = \{(-z, -g(-z)) : z \in D\}.
\end{aligned}$$

In this situation our problem boils down to the (non-)polynomial convexity of $D_1 \cup D_2$.

An appropriate tool in this context is Kallin's lemma: suppose X_1

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and X_2 are polynomially convex subsets of $\mathbb{C}^n$, suppose there is a polynomial p mapping X_1 and X_2 into two polynomially convex subsets Y_1 and Y_2 of the complex plane such that 0 is a boundary point of both Y_1 and Y_2 and with $Y_1 \cap Y_2 = \{0\}$. If $p^{-1}(0) \cap (X_1 \cup X_2)$ is polynomially convex, then $X_1 \cup X_2$ is polynomially convex, [1, 10].
In [3] Nguyen and the first author obtained a positive answer to our approximation question in a real-analytic situation for a new class of functions g. By using a biholomorphic change of coordinates, it is possible to assume that the first disk is the standard disk $\{(z, \bar{z}) : z \in D\}$ and then apply an approximation result of Nguyen, [2]. In the present paper the same idea of applying a biholomorphic map near the origin together with already developed techniques in this area is used. We obtain several new results of the form $[z^2, g^2; D] = C(D)$, one of them being a generalization of the main result of [3], for new and larger classes of functions g (theorem 2.5).

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2. An approximation result

We agree on the following convention: all functions defined in a neighborhood of the origin are of class C^1, even if we do not mention this explicitely.

Definition 2.1. Let $g(z)$ be an *even* function defined near the origin with $g(z) = o(z)$. Suppose that there exists a polynomial $p(\zeta_1, \zeta_2)$ such that for all functions $R(z)$ with $R(z) = o(g(z))$ both

$$\operatorname{Im} p(z, \bar{z} + g(z) + R(z)) > 0$$

and

$$\operatorname{Im} p(z, \bar{z} - g(z) + R(z)) < 0$$

hold for all $z \neq 0$ sufficiently close to 0.
Then we say that g satisfies the *polynomial condition* (with respect to p).

Examples 2.2.
- If $m > 1$, then for $g(z) = i|z|^m$ one can take $p(\zeta_1, \zeta_2) = \zeta_1 + \zeta_2$.
- For the function $g(z) = a|z|^2 + b\bar{z}^2$ with $|b| < |a|$ one can take $p(\zeta_1, \zeta_2) = -ia\zeta_1 + i\bar{a}\zeta_2$. From this fact a version of the main result of [5] follows.
- The function $g(z) = |z|^2 + \bar{z}^2$ does not satisfy the polynomial condition because it has non-zero zeroes.
- The function $g(z) = z^3\bar{z}$ satisfies the polynomial condition with respect to $p(\zeta_1, \zeta_2) = -i\zeta_1^3 + i\zeta_2^3$.

Lemma 2.3. *If g satisfies the polynomial condition with respect to a polynomial p, then g satisfies the polynomial condition with respect to the odd part of the polynomial p.*

Proof. Fix $R(z) = o(g(z))$ for the moment, then for $z \neq 0$ close to 0, we have:

(a) $$\operatorname{Im} p(z, \bar{z} + g(z) + R(z)) > 0,$$

(b) $$\operatorname{Im} p(z, \bar{z} - g(z) - R(-z)) < 0.$$

Replace z by $-z$ in (b) and use the fact that g is even, then also:

(c) $$\operatorname{Im} p(-z, -\bar{z} - g(z) - R(z)) < 0.$$

Now write p as a sum of homogeneous analytic polynomials, in other words $p = p_s + \cdots + p_n$ where p_j is homogeneous of degree j. Rewrite (c), for small $z \neq 0$, as:

$$\sum_{j=s}^{n} (-1)^{j+1} p_j(z, \bar{z} + g(z) + R(z)) > 0.$$

Combination with (a) shows that all terms with j even in (a) drop out. In a similar way these terms can be removed in the second part of the polynomial condition. □

We need the following lemma which is without doubt well-known.

Auxiliary lemma 2.4. *Let $F(w_1, w_2)$ be holomorphic near the origin, let $l \geq 2$ be an integer and let $F(w_1, w_2) = O(\|(w_1, w_2)\|^l)$.*
Let $A(w_1, w_2)$ be defined near the origin with

$$A(w_1, w_2) = O(\|(w_1, w_2)\|).$$

Then sufficiently close to the origin

$$F(w_1, w_2 + A(w_1, w_2)) = F(w_1, w_2) + A(w_1, w_2) B(w_1, w_2),$$

with $B(w_1, w_2) = O(\|(w_1, w_2)\|^{l-1})$.

Proof. As $F(w_1, w_2)$ is holomorphic near the origin,

$$H(w_1, w_2, w_3) = \begin{cases} \frac{F(w_1, w_3) - F(w_1, w_2)}{w_3 - w_2}, & \text{if } w_3 \neq w_2, \\ \frac{\partial F}{\partial \zeta_2}(w_1, w_2), & \text{if } w_3 = w_2, \end{cases}$$

is holomorphic near the origin, $H(w_1, w_2, w_3) = O(\|(w_1, w_2, w_3)\|^{l-1})$ and

$$F(w_1, w_2 + z) = F(w_1, w_2) + z H(w_1, w_2, w_2 + z).$$

Since $A(w_1, w_2) = O(\|(w_1, w_2)\|)$ it follows that

$$F(w_1, w_2 + A(w_1, w_2)) = F(w_1, w_2) + A(w_1, w_2) B(w_1, w_2),$$

and

$$B(w_1, w_2) = H(w_1, w_2, w_2 + A(w_1, w_2)) = O(\|(w_1, w_2)\|^{l-1}).$$

□

Theorem 2.5.
- *Let $F(w_1, w_2)$ be an odd holomorphic function near the origin satisfying $F(w_1, w_2) = O(\|(w_1, w_2)\|^3)$ and let $f(z) = F(z, \bar{z})$.*
- *Suppose that g satisfies the polynomial condition.*

• *Let h be defined near the origin with $h(z) = o(g(z))$.*
Then for all disks D about 0 with sufficiently small radius

$$[\ z^2,\ (\bar{z} + f(z) + g(z) + h(z))^2\ :\ D\ ] = C(D).$$

Proof. Let $X = \{\ (z^2,\ (\bar{z} + f(z) + g(z) + h(z))^2)\ :\ z \in D\ \}$.
The inverse image of X under the map $\Pi : \mathbb{C}^2 \to \mathbb{C}^2$, defined by $\Pi(\zeta_1, \zeta_2) = (\zeta_1^2, \zeta_2^2)$ consists of

$$\begin{aligned}
D_1 &= \{\ (z, \bar{z} + f(z) + g(z) + h(z))\ :\ z \in D\ \},\\
D_2 &= \{\ (-z, -(\bar{z} + f(z) + g(z) + h(z)))\ :\ z \in D\ \}\\
&= \{\ (z, \bar{z} + f(z) - g(z) - h(-z))\ :\ z \in D\ \},\\
D_3 &= \{\ (-z, \bar{z} + f(z) + g(z) + h(z))\ :\ z \in D\ \},\\
D_4 &= \{\ (z, -(\bar{z} + f(z) + g(z) + h(z)))\ :\ z \in D\ \}\\
&= \{\ (-z, \bar{z} + f(z) - g(z) - h(-z))\ :\ z \in D\ \}.
\end{aligned}$$

Note that the condition on the existence of the polynomial p implies that g has no non-zero zeroes and that the two functions z^2 and $(\bar{z} + f(z) + g(z) + h(z))^2$ separate the points of D (if D is sufficiently small).
The techniques developed in the papers [8, 5] on approximation on disks give us:

$$\begin{aligned}
&[\ z^2,\ (\bar{z} + f(z) + g(z) + h(z))^2\ :\ D\ ] = C(D)\\
\Longleftrightarrow\ & P(X) = C(X)\\
\Longleftrightarrow\ & X \text{ is polynomially convex}\\
\Longleftrightarrow\ & D_1 \cup D_2 \cup D_3 \cup D_4 \text{ is polynomially convex}\\
\Longleftrightarrow\ & D_1 \cup D_2 \text{ is polynomially convex.}
\end{aligned}$$

We comment on these equivalences. The first equivalence is trivial. Since X is totally real except at the origin, the second one follows from a theorem of O'Farrell, Preskenis and Walsh, [4]. The next equivalence is a consequence of a theorem of Sibony, [11], and the last one is an application of Kallin's lemma using the polynomial $p(\zeta_1, \zeta_2) = \zeta_1 \cdot \zeta_2$.
Later on we will also use the following theorem of Wermer, [12]. *If the function F is of class C^1 near the origin in the complex plane, with $F_{\bar{z}}(0) \neq 0$, then $[z, F : D] = C(D)$ if D is a sufficiently small disk around 0.* This implies that all disks D_i are polynomially convex.
For precise statements and use of these theorems, see [8], in particular the proof of theorem 1.

Now let us show that $D_1 \cup D_2$ is polynomially convex. Consider the map $G(w_1, w_2) = (w_1, w_2 + F(w_1, w_2))$. Since $F(w_1, w_2) = O(\|(w_1, w_2)\|^3)$ it follows that G is biholomorphic near the origin (with inverse called H).
Now $E_1 = H(D_1)$ consists of points of the form $(z, q(z))$ where q is of class C^1 near 0 and $q(0) = 0$. Then there are a and b such that $q(z) = az + b\bar{z} + r(z)$, where $r(z) = o(z)$. Applying G we see

$$(*) \qquad (z, q(z) + F(z, q(z))) = (z, \bar{z} + f(z) + g(z) + h(z)).$$

Since $f(z)+g(z)+h(z) = O(z^3)+o(z)+o(z)$ and moreover $F(z,q(z)) = O(z^3)$ we infer that $q(z) = \bar{z} + r(z)$. So $(*)$ translates into

$$(z, \bar{z} + r(z) + F(z, \bar{z} + r(z))) = (z, \bar{z} + f(z) + g(z) + h(z)).$$

Applying the auxiliary lemma to this expression with $w_1 = z, w_2 = \bar{z}$ and $A(w_1, w_2) = r(w_1)$ we obtain:

$$(z, \bar{z} + r(z) + f(z) + r(z)B(z, \bar{z})) = (z, \bar{z} + f(z) + g(z) + h(z)).$$

It follows that

$$r(z) = \frac{g(z) + h(z)}{1 + B(z, \bar{z})} = g(z) + \frac{h(z) - g(z)B(z, \bar{z})}{1 + B(z, \bar{z})}.$$

We conclude that $E_1 = H(D_1)$ consists of points $(z, \bar{z} + g(z) + R_1(z))$ in which $R_1(z) = o(g(z))$ and is of class C^1. This last fact follows from the definition of $B(w_1, w_2)$ in the proof of the auxiliary lemma.
Now E_1 is polynomially convex if D is sufficiently small (Wermer).
Similarly $E_2 = H(D_2)$ consists of points $(z, \bar{z} - g(z) + R_2(z))$ in which $R_2(z) = o(g(z))$ and is of class C^1. Also E_2 is polynomially convex if D is sufficiently small. Since g satisfies the polynomial condition, Kallin's lemma can be applied, showing that $E_1 \cup E_2$ is polynomially convex. Applying G it follows that $D_1 \cup D_2$ is polynomially convex for sufficiently small D. $\square$

Remark 2.6. If $F(w_1, w_2) = f(w_1) = O(w_1^3)$ no computation is necessary since the map

$$G(w_1, w_2) = (w_1, w_2 + f(w_1))$$

has inverse $H(z_1, z_2) = (z_1, z_2 - f(z_1))$ near the origin. We now obtain directly $H(z, \bar{z}+f(z)+g(z)+h(z)) = (z, \bar{z}+g(z)+h(z))$ and similarly $H(z, \bar{z} + f(z) - g(z) - h(-z)) = (z, \bar{z} - g(z) - h(-z))$. Now use that g satisfies the polynomial condition and proceed as before.

3. The polynomial condition for homogeneous functions

Let g satisfy the polynomial condition, then there is an odd polynomial p such that

$$\operatorname{Im} p(z, \bar{z} + g(z) + R(z)) > 0 \tag{1}$$

and

$$\operatorname{Im} p(z, \bar{z} - g(z) + R(z)) < 0 \tag{2}$$

hold for all $z \neq 0$ sufficiently close to 0 if $R(z) = o(g(z))$. As before g is even, but instead of $g(z) = o(z)$ we impose a stronger condition on this function:

$$g \text{ is } \textit{homogeneous} \text{ of degree } m > 1, \text{ i.e.}$$

$$g(tz) = t^m g(z) \quad \text{for } t > 0$$

(so in fact g is defined everywhere). Now write p as a sum of homogeneous analytic polynomials, $p = p_{2s-1} + \cdots + p_{2n-1}$ where all p_k are homogeneous of odd degree k. We assume first that m is not an odd

integer. Let $n_0 \le n$ be maximal such that $2n_0 - 1 < 2s - 2 + m$. Taking for R the zero function we obtain:

$$p(z, \bar{z} + g(z)) = p_{2s-1}(z,\bar{z}) + \cdots + p_{2n_0-1}(z,\bar{z}) \\ + \frac{\partial p_{2s-1}}{\partial \zeta_2}(z,\bar{z}) \cdot g(z) + O(|z|^{\alpha}),$$

for some $\alpha > 2s - 2 + m$. Now we restrict z to the unit circle Γ, and obtain for $t > 0$:

$$p(tz, t\bar{z} + g(tz)) = t^{2s-1} p_{2s-1}(z,\bar{z}) + \cdots + t^{2n_0-1} p_{2n_0-1}(z,\bar{z}) \\ + t^{2s-2+m} \frac{\partial p_s}{\partial \zeta_2}(z,\bar{z}) \cdot g(z) + O(t^{\alpha}).$$

Now take the imaginary part, divide by t^{2s-1} and let t tend to 0. We obtain $\operatorname{Im} p_{2s-1}(z,\bar{z}) \ge 0$. Similarly, using the second condition on g, we obtain $\operatorname{Im} p_{2s-1}(z,\bar{z}) \le 0$, hence $\operatorname{Im} p_{2s-1}(z,\bar{z}) = 0$ for all $z \in \Gamma$ (hence for all $z \in \mathbb{C}$). Writing $p_{2s-1}(\zeta_1,\zeta_2) = \sum_{k=0}^{2s-1} a_k \zeta_1^k \zeta_2^{2s-1-k}$ this means that $a_k = \overline{a_{2s-1-k}}$ for all $k = 0, \ldots, 2s-1$. We call such a polynomial *complex-symmetric.*
Repeating this reasoning we successively obtain:

$$\operatorname{Im} p_{2s+1}(z,\bar{z}) = 0, \ldots, \operatorname{Im} p_{2n_0-1}(z,\bar{z}) = 0$$

and

$$(*) \qquad \operatorname{Im} \frac{\partial p_{2s-1}}{\partial \zeta_2}(z,\bar{z}) \cdot g(z) \ge 0.$$

Also in the case that m is an odd integer (1) and (2) in a similar way as above lead to $(*)$.

Now suppose that for all $z \in \Gamma$ the inequality $(*)$ is strict then we will show that the polynomial condition is satisfied for g with respect to the polynomial p_{2s-1}. Indeed, if $R(z) = o(g(z))$ it follows for small $z \ne 0$:

$$p_{2s-1}(z, \bar{z} + g(z) + R(z)) \\ = p_{2s-1}(z,\bar{z}) + \frac{\partial p_{2s-1}}{\partial \zeta_2}(z,\bar{z}) \cdot g(z) \cdot \big(1 + \frac{R(z)}{g(z)}\big) + O(|z|^{2s-3+2m}).$$

So for $z \in \Gamma$ and small $t > 0$ it follows that:

$$\operatorname{Im} p_{2s-1}(tz, t\bar{z} + g(tz) + R(tz)) \\ = \operatorname{Im} t^{2s-2+m} \Big(\frac{\partial p_{2s-1}}{\partial \zeta_2}(z,\bar{z}) \cdot g(z) \cdot \big(1 + \frac{R(tz)}{g(tz)}\big) + O(t^{m-1}) \Big).$$

Since $\frac{R(tz)}{g(tz)}$ is uniformly small on Γ if $t > 0$ is sufficiently small, the above expression is positive on Γ for small $t > 0$. In other words: $\operatorname{Im} p_{2s-1}(z, \bar{z} + g(z) + R(z)) > 0$ if $z \ne 0$ is sufficiently small. Also $\operatorname{Im} p_{2s-1}(z, \bar{z} - g(z) + R(z)) < 0$ for small $z \ne 0$. So g satisfies the polynomial condition with respect to p_{2s-1} and we proved:

Theorem 3.1. *If g is even and of class C^1 near the origin in the complex plane, is homogeneous of order $m > 1$ and satisfies*

$$Im \frac{\partial p_{2s-1}}{\partial \zeta_2}(z, \bar{z}) \cdot g(z) > 0 \quad \textit{for all} \;\; z \in \Gamma,$$

where p_{2s-1} is a homogeneous complex-symmetric polynomial of degree $2s-1$, then g satisfies the polynomial condition with respect to p_{2s-1}.

Example 3.2. An example of such a function is $g(z) = i\,\overline{\frac{\partial p_{2s-1}}{\partial \zeta_2}(z, \bar{z})}$, where p_{2s-1} is any homogeneous complex-symmetric polynomial of degree $2s-1 \geq 3$ ($s = 1$ excluded because g has to be homogeneous of degree $m > 1$) and such that $\frac{\partial p_{2s-1}}{\partial \zeta_2}(z, \bar{z})$ has no non-zero zeroes.

Theorem 3.3. *Let $g(z) = \sum_{k=-\infty}^{\infty} a_k \bar{z}^k z^{2m-k}$ with m a positive integer. Suppose that $\sum_{k=-\infty}^{\infty} |ka_k| < \infty$ and that one of the following increasingly weaker conditions is met:*

$$\exists\; l \leq m \;\; \textit{such that} \;\; |a_l| > \sum_{n \neq l} |a_n|,$$

or

$$\exists\; l \leq m \;\; \textit{such that} \;\; \sum_{n=1}^{\infty} \Big|\frac{a_{l+n}}{a_l} + \frac{\bar{a}_{l-n}}{\bar{a}_l}\Big| < 1,$$

or

$$\exists\; l \leq m \;\; \textit{such that} \;\; \mathrm{Re}\Big(1 + \sum_{n=1}^{\infty} \big(\frac{a_{l+n}}{a_l} + \frac{\bar{a}_{l-n}}{\bar{a}_l}\big) w^n\Big) > 0 \;\; \textit{on} \;\; |w| = 1.$$

Then g is an even homogeneous C^1 function of degree $2m$ that satisfies the polynomial condition.

Proof. Let $p(\zeta_1, \zeta_2) = \bar{\alpha}\zeta_1^{2m-2l+1} + \alpha\zeta_2^{2m-2l+1}$ with α to be determined later (and with $l \leq m$). Then for $z \in \Gamma$:

$$\frac{1}{2m-2l+1} \mathrm{Im} \frac{\partial p}{\partial \zeta_2}(z, \bar{z}) \cdot g(z) = \mathrm{Im} \sum_{k=-\infty}^{\infty} \alpha a_k \bar{z}^{2m-2l+k} z^{2m-k}$$

$$= \mathrm{Im}\Big\{ \sum_{k=-\infty}^{l-1} \alpha a_k \bar{z}^{2m-2l+k} z^{2m-k} + \alpha a_l |z|^{4m-2l} + \sum_{k=l+1}^{\infty} \alpha a_k \bar{z}^{2m-2l+k} z^{2m-k} \Big\}$$

$$= \mathrm{Im}\Big\{ \alpha a_l |z|^{4m-2l} + \sum_{n=1}^{\infty} \big(\alpha a_{l+n} - \bar{\alpha}\bar{a}_{l-n}\big) \bar{z}^{2m-l+n} z^{2m-l-n} \Big\}$$

$$= \mathrm{Im}\Big\{ i|a_l||z|^{4m-2l} \Big(1 + \sum_{n=1}^{\infty} \big(\frac{a_{l+n}}{a_l} + \frac{\bar{a}_{l-n}}{\bar{a}_l}\big)\big(\frac{\bar{z}}{z}\big)^n\Big) \Big\}.$$

In the last equality we chose $\alpha = i\frac{|a_l|}{a_l}$. The final expression has positive imaginary part if the third condition in the statement of the theorem is satisfied. $\square$

Remarks 3.4. This result includes the more restricted case of polynomials $g(z) = \sum_{k=0}^{2m} a_k \bar{z}^k z^{2m-k}$ in z and $\bar{z}$, for which there exists $0 \leq l \leq m$ such that $|a_l| > \sum_{k \neq l} |a_k|$, essentially studied by Nguyen,

[2], and applied in a real-analytic setting by Nguyen and De Paepe, [3]. The condition on the coefficients here is more general. For instance if $m = 1$ the condition is valid if $\left|\frac{a_2}{a_1} + \frac{\bar{a}_0}{\bar{a}_1}\right| < 1$, which is certainly the case for (but is not equivalent to) $|a_1| > |a_0| + |a_2|$.

Example 3.5. Applying theorem 3.3 and theorem 2.5 we obtain a result from [9]:

$$[\, z^2,\ \bar{z}^2 + z^3;\ D\,] = [\, z^2,\ (\bar{z} + \frac{1}{2}\frac{z^3}{\bar{z}} + \text{h.o.t.})^2\,;\ D\,] = C(D).$$

4. Another use of a biholomorphic map

In theorem 2.5 it was fruitful to apply a biholomorphic map in order to show polynomial convexity. This idea can be used in other situations as well. For instance, suppose that g is of class C^1 near 0, $g(0) = 0, g_z(0) = 0, g_{\bar{z}}(0) = 1$ and such that z^2 and g^2 separate points near 0. Also suppose F is defined near the origin, holomorphic, and odd, with $F(w_1, w_2) = O(\|(w_1, w_2)\|^3)$. Then z^2 and $(g+F(z,g))^2$ separate points near 0 and $[\, z^2,\ (g + F(z,g))^2\,;\ D\,] \subset [\, z^2,\ g^2\,;\ D\,]$. So $[\, z^2,\ g^2; D\,] \neq C(D)$ implies $[\, z^2,\ (g + F(z,g))^2; D\,] \neq C(D)$. This is the contents of the proof of theorem 2 in [9]. But more is true.

Theorem 4.1. *With notation as above and for sufficiently small D:*

$$[\, z^2,\ g^2; D\,] = C(D) \iff [\, z^2,\ (g + F(z,g))^2; D\,] = C(D).$$

Proof. Let $X = \{\, (z^2,\ g(z)^2)\ :\ z \in D\,\}$, furthermore if we let $Y = \{\, (z^2,\ (g(z) + F(z, g(z)))^2)\ :\ z \in D\,\}$, then, using the biholomorphic map $G(w_1, w_2) = (w_1, w_2 + F(w_1, w_2))$ in the fourth equivalence, we obtain for sufficiently small D:

$$
\begin{aligned}
& [\, z^2,\ g^2; D\,] = C(D) \\
\iff & P(X) = C(X) \\
\iff & X \text{ is polynomially convex} \\
\iff & \{\, (z, g(z))\ :\ z \in D\,\} \cup \{\, (z, -g(-z))\ :\ z \in D\,\} \text{ is pcx} \\
\iff & \{\, (z, g(z) + F(z, g(z)))\ :\ z \in D\,\} \\
& \qquad \cup \{\, (z, -g(-z) + F(z, -g(-z)))\ :\ z \in D\,\} \text{ is pcx} \\
\iff & Y \text{ is polynomially convex} \\
\iff & P(Y) = C(Y) \\
\iff & [\, z^2,\ (g + F(z,g))^2 : D\,] = C(D).
\end{aligned}
$$

$\square$

Question 4.2. Is $[\, z^2,\ (g + F(z,g))^2\ :\ D\,] = [\, z^2,\ g^2\,;\ D\,]$ for all g and D as above?

Example 4.3. In the case $F(w_1, w_2) = f(w_1) = O(w_1^3)$ the answer to the question is yes:

$$[\, z^2,\ (g + f)^2\ :\ D\,] = [\, z^2,\ g^2\ :\ D\,].$$

Indeed, since zf, f^2 and $\frac{f}{z}$ are even analytic functions, they belong to $A = [\, z^2,\ (g + f)^2\ :\ D\,]$. Also $z^2(g + f)^2 \in A$, thus (since the real part of $z^2(g + f)^2$ is non-negative near the origin) $z(g + f) \in A$, hence $zg \in A$. Also $(g + f)^2 = g^2 + 2(zg) \cdot \frac{f}{z} + f^2 \in A$, therefore $g^2 \in A$. Hence $A = [\, z^2,\ g^2\ :\ D\,]$.

Example 4.4. A second situation where the answer is yes occurs when $F(w_1, w_2)$ has the form $w_2 G(w_1^2, w_2^2)$ where G is holomorphic near the origin with $G(0,0) = 0$. Then $(g + F(z,g))^2$ can be written as $g^2 + g^2 H(z^2, g^2)$ with $H(0,0) = 0$. The map

$$(w_1, w_2) \mapsto (w_1, w_2 + w_2 H(w_1, w_2))$$

is biholomorphic near the origin and maps the pair (z^2, g^2) to $(z^2, (g + F(z,g))^2)$. This shows that the algebra generated by z^2 and g^2 on a small D equals the algebra generated by z^2 and $(g + F(z,g))^2$ on D.

5. Appendix

In this appendix we keep the setting of section 3 and see what can be said when $\operatorname{Im} \frac{\partial p_{2s-1}}{\partial \zeta_2}(z, \bar{z}) \cdot g(z)$ has zeroes on Γ. Under stronger conditions on F, g, and h, we obtain the following approximation result.

Theorem 5.1. *Let $F(w_1, w_2)$ be an odd holomorphic function near the origin satisfying $F(w_1, w_2) = O(\|(w_1, w_2)\|^5)$ and set $f(z) = F(z, \bar{z})$. Suppose that the following conditions are met:*

- *The function $g \in C^1$ is even and homogeneous of degree $m > 3$.*
- *There are homogeneous complex-symmetric polynomials p_{2s-1} and p_{2s+1} of degree $2s-1$, respectively $2s+1$, such that*

$$\operatorname{Im} \frac{\partial p_{2s-1}}{\partial \zeta_2}(z, \bar{z}) \cdot g(z) \geq 0, \quad \textit{for all } z \in \Gamma,$$

  and

$$\operatorname{Im} \frac{\partial p_{2s+1}}{\partial \zeta_2}(z, \bar{z}) \cdot g(z) > 0$$

  for all $z \in \Gamma$ where $\operatorname{Im} \frac{\partial p_{2s-1}}{\partial \zeta_2}(z, \bar{z}) \cdot g(z) = 0$.
- *h is defined near the origin and $h(z) = o(z^2 g(z))$.*

Then for all disks D centered at 0 with sufficiently small radius

$$[\, z^2,\ (\bar{z} + f(z) + g(z) + h(z))^2\ ;\ D\,] = C(D).$$

Proof. We will follow the line of the proofs of the auxiliary lemma 2.4 and of theorem 2.5, as well as the notation. We see that $B(z, \bar{z}) = O(z^4)$ since $F(w_1, w_2) = O(\|(w_1, w_2)\|^5)$. From this fact and $h(z) = o(z^2 g(z))$ it follows that $R_1(z), R_2(z) = o(z^2 g(z))$.

Let N be the set of points $z \in \Gamma$ where

$$\operatorname{Im} \frac{\partial p_{2s-1}}{\partial \zeta_2}(z, \bar{z}) \cdot g(z) = 0.$$

Now assume

$$\operatorname{Im} \frac{\partial p_{2s+1}}{\partial \zeta_2}(z, \bar{z}) \cdot g(z) > 0 \text{ for all } z \in N.$$

Then there is $\lambda_0 > 0$ and $\delta > 0$ such that for all $z \in \Gamma$ and $0 < \lambda \leq \lambda_0$:

$$\operatorname{Im} \frac{\partial \big(p_{2s-1} + \lambda p_{2s+1}\big)}{\partial \zeta_2}(z, \bar{z}) \cdot g(z) \geq \lambda \delta.$$

Indeed, for $z \in \Gamma$, let

$$f_0(z) = \operatorname{Im} \frac{\partial p_{2s-1}}{\partial \zeta_2}(z, \bar{z}) \cdot g(z), \qquad f_1(z) = \operatorname{Im} \frac{\partial p_{2s+1}}{\partial \zeta_2}(z, \bar{z}) \cdot g(z).$$

Let $0 < 2\delta = \inf_{z \in N} f_1(z)$, U a neighbourhood of N in Γ such that $\inf_{z \in U} f_1(z) \geq \delta$ and $\epsilon = \inf_{z \in \Gamma \setminus U} f_0(z) > 0$.
If we take $0 < \lambda \leq \lambda_0 = \min\{\frac{\epsilon/2}{\|f_1\|_\Gamma}, \frac{\epsilon}{2\delta}\}$, then $f_0 + \lambda f_1 \geq \lambda\delta$ on Γ.

Now for $m > 3$ and $R(z) = o(z^2 g(z))$ we have:

$$(p_{2s-1} + p_{2s+1})(z, \bar{z} + g(z) + R(z)) = p_{2s-1}(z, \bar{z}) + p_{2s+1}(z, \bar{z})$$
$$+ \big(\frac{\partial p_{2s-1}}{\partial \zeta_2}(z, \bar{z}) + \frac{\partial p_{2s+1}}{\partial \zeta_2}(z, \bar{z})\big) \cdot g(z) \cdot \big(1 + \frac{R(z)}{g(z)}\big) + O(|z|^{2s-3+2m}).$$

So for $z \in \Gamma$ and small $t > 0$ one has

$$\begin{aligned} &\operatorname{Im}(p_{2s-1} + p_{2s+1})(tz, t\bar{z} + g(tz) + R(tz)) \\ &\quad = t^{2s-2+m} \operatorname{Im} \big[\big(\frac{\partial p_{2s-1}}{\partial \zeta_2}(z, \bar{z}) + t^2 \frac{\partial p_{2s+1}}{\partial \zeta_2}(z, \bar{z})\big) \\ &\qquad \cdot g(z) \cdot \big(1 + t^2 \cdot \frac{z^2 R(tz)}{(tz)^2 g(tz)}\big) + O(t^{m-1})\big] \geq \frac{1}{2} t^{2s+m} \delta, \end{aligned}$$

since $\frac{z^2 R(tz)}{(tz)^2 g(tz)}$ is uniformly small on Γ if $t > 0$ is sufficiently small.
So $\operatorname{Im}(p_{2s-1} + p_{2s+1})(z, \bar{z} + g(z) + R(z)) > 0$, and similarly $\operatorname{Im}(p_{2s-1} + p_{2s+1})(z, \bar{z} - g(z) + R(z)) < 0$ if z is sufficiently small. Now proceed as in the proof of theorem 2.5. $\square$

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